ON THE PROPAGATION OF STRONG DISCONTINUITIES IN A MULTI-COMPONENT MEDIUM

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The motion of a multi-component medium is considered as consisting of the mutually permeating motions of the components which make up the medium, as was proposed by Rakhmatulin [1]. The motion of each component is analogous to motion in a porous medium. Along with the concept of the true density of the n'th component $\rho_n{}^0$ we make use also of the concept of the peculiar density $\rho_n{}^0 = M_n/W$ of the n'th component. Here M_n is the mass of the n'th component in a volume W of the medium.

Consideration is limited to those media in which the pressure at each point can be taken as common to all the components.

1. Relations at a strong discontinuity. Suppose that a strong discontinuity surface is propagated in a space occupied by a mixture consisting of N components. Let us consider an element dS of the surface of discontinuity in a fixed rectangular system of coordinates, with the axis of y directed parallel to the normal to the element under consideration at a given instant of time t. Let us denote by u_n, v_n, w_n the projections of the velocity V_n of particles of the n'th component on the coordinate axes, and by D the velocity of translation of the element along the normal to the front.

We shall denote the parameters of the medium in front of and behind the discontinuity surface by the subscripts + and − respectively. We shall specify an element of volume of the medium $(D - v_{n+})dSdt$ in front of the element dS of the discontinuity surface at time t. Then in the interval dt all the particles of the n'th component contained in the specified volume element at the time t will pass behind the discontinuity surface.

We shall now specify an element of volume of the medium $(D - v_{n-})dSdt$ behind the element dS of the surface of discontinuity, containing the same particles of the n'th component. It is clear that the mass of the n'th component in both the specified volume elements is the same, although the masses of the remaining $N - 1$ components will, generally speaking, be

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different. The law of conservation of mass of the n'th component at the strong discontinuity can be written in the form

$$(D - v_{n+})\rho_{n+} = (D - v_{n-})\rho_{n-} \qquad (n = 1, \ldots, N) \qquad (1.1)$$

where ρ_{n+}, ρ_{n-} are the peculiar densities of the n'th component before and behind the discontinuity surface.

Let us write down the equations for the conservation of momentum along the coordinate axes for the specified mass of the n'th component. Bearing in mind that the force due to the pressure p on any element of area dS, when apportioned between the components of the mixture, amounts to a share of $p(\rho_n/\rho_n^0)dS$ for the n'th component, we obtain

$$\begin{aligned} &\rho_{n+}(D - v_{n+})^2 + p_+\rho_{n+}/\rho_{n+}^\circ = \rho_{n-}(D - v_{n-})^2 + p_-\rho_{n-}/\rho_{n-}^\circ \\ &\rho_{n+}(D - v_{n+})u_{n+} = \rho_{n-}(D - v_{n-})u_{n-} \qquad (n = 1, \ldots, N) \\ &\rho_{n+}(D - v_{n+})w_{n+} = \rho_{n-}(D - v_{n-})w_{n-} \end{aligned} \qquad (1.2)$$

The forces of interaction of the n'th component with the remaining components do not appear in the equations so obtained, in so far as these forces are infinitely small.

The system of equations (1.1), (1.2) will be supplemented by the relations

$$p_- = \varphi_n(\rho_{n-}^\circ, p_0, \rho_{0n}^\circ) \qquad (n = 1, \ldots, N) \qquad (1.3)$$

which we shall assume to be part of the data, and by the relation

$$\sum_{n=1}^{N} \frac{\rho_{n-}}{\rho_{n-}^\circ} = 1 \qquad (1.4)$$

arising from the fact that the ratio ρ_{n-}/ρ_{n-}^0 is the fraction of the volume of the medium occupied by the n'th component.

It must be borne in mind that similar relations apply to the parameters of the components in front of the discontinuity surface:

$$\sum_{n=1}^{N} \frac{\rho_{n+}}{\rho_{n+}^\circ} = 1, \qquad p_+ = \varphi_n(\rho_{n+}^\circ, p_0, \rho_{0n}^\circ) \qquad (1.5)$$

The quantities p_0, ρ_{0n}^0 are certain initial values of the pressure and density and, in particular, may be taken as

$$p_0 = p_+, \quad \rho_{0n}^\circ = \rho_{n+}^\circ$$

Let us consider the possible types of discontinuity in the multi-component medium. In the case when there is no flow of the medium across the surface of discontinuity, we have

$$v_{n+} = v_{n-} = D \qquad (n = 1, \ldots, N)$$

Then equation (1.1) and also the last two equations of the system (1.2) are satisfied by arbitrary values ρ_{n+}, ρ_{n-}, u_{n+}, u_{n-}, w_{n+}, w_{n-}, and the first of the equations (1.2) gives

$$p_-\rho_{n-}/\rho_{n-}{}^\circ = p_+\rho_{n+}/\rho_{n+}{}^\circ \qquad (n = 1, \ldots, N)$$

Adding these relations for all N components and using (1.4) and the first of equations (1.5), we obtain $p_- = p_+$. Then it follows from the last relation that

$$\rho_{n-}/\rho_{n-}{}^\circ = \rho_{n+}/\rho_{n+}{}^\circ \qquad (n = 1, \ldots, N) \tag{1.6}$$

If the projections of the velocities, tangent to the surface of discontinuity, are the same on both sides of the discontinuity: $u_{n+} = u_{n-}$, $w_{n+} = w_{n-} (n = 1, \ldots, N)$ and the true densities of all the N components or certain of them (N_1) are different: $\rho_{n-}{}^0 \neq \rho_{n+}{}^0 (n = 1, \ldots, N_1)$, then by analogy with the case of a single-component medium the discontinuity can be called a contact discontinuity in the multi-component medium. In this case the peculiar densities of the respective components are also different and they satisfy the relations (1.6).

If the values of at least one of the projections of the tangential velocities are not equal on the two sides of the discontinuity, then there is a tangential discontinuity in the multi-component medium. Then the densities may be either different or the same.

The discontinuities so far considered are completely analogous with the corresponding discontinuities in a single-component medium. In contrast, however, to a single-component medium, a mixture can permit the propagation of discontinuities which, being tangential (with regard to certain components), are nevertheless at the same time shock waves. In fact, suppose that there is no flow of certain components (N_1) across the discontinuity surface; then $v_{n+} = v_{n-} = D (n = 1, \ldots, N_1)$ and by virtue of the last two equations of (1.2) the inequalities $u_{n-} \neq u_{n+}$, $w_{n-} \neq w_{n+}$ $(n = 1, \ldots, N_1)$ may obtain as before for the N_1 components under consideration, i.e. we have a tangential discontinuity. However, in so far as there is a flow of the remaining components across the given surface of discontinuity, then this surface constitutes a shock wave and, in particular, we may show that $p_- \neq p_+$. Discontinuities of this type are called complex discontinuities.

Let us consider, for example, the case of the motion of a two-component medium, in which there is a flow of the first component, but not of the second, across the surface of discontinuity. Then from the system (1.1), (1.2), (1.3), (1.4), written out for this case, the following relations can be obtained:

$$(v_{2+} - v_{1+})^2 = \frac{p_- - p_+}{\rho_{1+}(1 - \rho_{1+}/\rho_{1-})} \tag{1.7}$$

$$\rho_{1-} = (1 - \varkappa)\rho_{1-}{}^\circ, \qquad \rho_{2-} = \varkappa\rho_{2-}{}^\circ \qquad \left(\varkappa = \frac{p_+\rho_{2+}}{p_-\rho_{2+}{}^\circ}\right) \tag{1.8}$$

$$(v_{2+} - v_{1+})^2 = \frac{(p_- - p_+)(1-\varkappa)}{\rho_{1+}(1-\varkappa-\rho_{1+}/\rho_{1-}^{\circ})} \tag{1.9}$$

$$v_{1-} - v_{1+} = (v_{2+} - v_{1+})\,\frac{1-\varkappa-\rho_{1+}/\rho_{1-}^{\circ}}{1-\varkappa} \tag{1.10}$$

It is obvious from (1.9) that to any value of $p_- > p_+$ there corresponds a positive value of $(v_{2+} - v_{1+})^2$ and consequently for any pressure $p_- > p_+$ we can find a value for the difference of the velocities of the components in front of the surface of discontinuity $|v_{2+} - v_{1+}|$, for which complex discontinuities can propagate. For increasing p_- the quantity $(v_{2+} - v_{1+})^2$ increases, moreover there exists a certain minimal value for the modulus of the difference in velocities of the components $|v_{2+} - v_{1+}|_{\min}$ for which the existence of a complex discontinuity is possible. This value is obtained from (1.9) by letting $p_- \to p_+$ and elucidating the indeterminacy of the type 0/0; then we have

$$(v_{2+} - v_{1+})_{\min} = \pm\left(\frac{p_+\rho_{2+}^{\circ}\rho_{1+}^{\circ}}{\rho_{2+}\rho_{1+}^{02} + \rho_{1+}\rho_{2+}^{\circ}p_+/a_{1+}^2}\right)^{1/2} \tag{1.11}$$

where $a_{1+}^2 = \phi_1'(\rho_{1+}^0, p_0, \rho_{01}^0)$ is a known function since the equation of state $p = \phi_1(\rho_1^0, p_0, \rho_{010})$ is given.

In the case when $|v_{2+} - v_{1+}| < |v_{2+} - v_{1+}|_{\min}$ the existence of a complex discontinuity becomes impossible. From (1.10) it is evident that in the case $v_{2+} > v_{1+}$ ($v_{2+} < v_{1+}$) we shall have

$$v_{1-} > v_{1+}\ (v_{1-} < v_{1+})$$

In so far as $p_-/p_+ > \rho_{2-}^0/\rho_{2+}^0$, then it follows from the second expression of (1.8) that

$$\rho_{2-} < \rho_{2+}$$

From (1.7) it is obvious that $\rho_{1-} > \rho_{1+}$, since otherwise the quantity $v_{2+} - v_{1+}$ would be imaginary.

Yet another peculiarity of the mixture in comparison with the single-component medium is the possibility[1] in principle of the existence in it of discontinuities, on both sides of which the pressure and true densities of all the components are the same, and yet there exists a flow of the components across the surface of discontinuity. In this case the peculiar densities and the velocities of motion will have discontinuities.

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[1] The real possibility of the existence of this discontinuity in a stream for any appreciable interval of time is bound up with the difficulties of practical achievement of all the boundary conditions which are necessary (see below) for its existence.

Let us restrict ourselves for the sake of simplicity to the case of equality of the velocities of all the components in front of the surface of discontinuity ($v_{n+} = v_+$, $n = 1, \ldots, N$). Then, adding the first equations of system (1.2) for all N components and using (1.1), (1.4) and the first equation (1.5), we obtain with $p_- = p_+$ the relation

$$\sum_{n=1}^{N} \rho_{n+}(v_{n-} - v_+) = 0$$

We see that for some of the components there is an increase of velocity at the discontinuity surface, and for other components a decrease.

For a two-component medium it is easy to obtain from (1.1), (1.2), (1.4) the following relations:

$$\begin{aligned} v_{1-} - v_+ &= \pm \frac{\rho_{2+}}{\rho_{1+}^\circ}\left(1 - \frac{\rho_{1+}^\circ}{\rho_{2+}^\circ}\right)\sqrt{\frac{p_+}{\rho_{1+} + \rho_{2+}}} \\ v_{2-} - v_+ &= \mp \frac{\rho_{1+}}{\rho_{1+}^\circ}\left(1 - \frac{\rho_{1+}^\circ}{\rho_{2+}^\circ}\right)\sqrt{\frac{p_+}{\rho_{1+} + \rho_{2+}}} \\ D - v_+ &= \mp \sqrt{\frac{p_+}{\rho_{1+} + \rho_{2+}}} \\ \rho_{1-} - \rho_{1+} &= -\frac{\rho_{2+}\rho_{1+}}{\rho_{2+} + \rho_{1+}}\left(1 - \frac{\rho_{1+}^\circ}{\rho_{2+}^\circ}\right) \\ \rho_{2-} - \rho_{2+} &= \frac{\rho_{2+}\rho_{1+}\rho_{2+}^\circ}{\rho_{1+}^\circ(\rho_{2+} + \rho_{1+})}\left(1 - \frac{\rho_{1+}^\circ}{\rho_{2+}^\circ}\right) \end{aligned} \tag{1.12}$$

For the sake of definiteness suppose that $\rho_{2+}^0 > \rho_{1+}^0$. Then $\rho_{1-} - \rho_{1+} < 0$, $\rho_{2-} - \rho_{2+} > 0$. Accordingly, at a discontinuity surface of equal pressure, the peculiar density of the denser component will increase, whilst that of the less dense component will decrease. An increase in velocity of the first component corresponds to a decrease in velocity of the second component and vice versa.

From the examples considered it can be observed that the velocities and peculiar densities of the components in the case $p_- \geqslant p_+$, $\rho_{n-}^0 \geqslant \rho_{n+}^0$ ($n = 1, \ldots, N$) can either increase or decrease at a surface of discontinuity in a multi-component medium, although the true densities of both components either increase or remain unaltered.

From what follows it is clear that this is true, not merely for the particular cases of discontinuities studied above.

Let us pass on now to consideration of shock waves in a multi-component medium, i.e. those discontinuities for which there is a flow of all the components across the surface of discontinuity, and moreover $p_- > p_+$.

Then it follows from the last two equations of (1.2) that

$$u_{n-} = u_{n+}, \qquad w_{n-} = w_{n+} \qquad (n = 1, \ldots, N)$$

From the system of the remaining $3N + 1$ equations (1.1), (1.2), (1.3), (1.4)

$$(D - v_{n+})\rho_{n+} = (D - v_{n-})\rho_{n-}$$

$$\rho_{n+}(D - v_{n+})^2 + p_+\rho_{n+}/\rho_{n+}^\circ = \rho_{n-}(D - v_{n-}) + p_-\rho_{n-}/\rho_{n-}^\circ$$

$$p_- = \varphi_n(\rho_{n-}^\circ, \ p_0, \ \rho_{0n}^\circ) \quad (n = 1, \ldots, N) \tag{1.13}$$

$$\sum_{n=1}^{N} \frac{\rho_{n-}}{\rho_{n-}^\circ} = 1$$

we can determine the remainder of the $3N + 1$ parameters of the mixture behind the surface of discontinuity (p_-, ρ_{n-}, ρ_{n-}^0, v_{n-}; $n = 1, \ldots, N$) and the velocity of translation of the discontinuity, if we are given a boundary condition which is equivalent to stipulating any one of the parameters behind the surface of discontinuity.

However, in the calculation of the motion of a multi-component medium problems often arise in which the boundary condition is equivalent to stipulating several of the parameters (the piston problem, the problem of the penetration of bodies into a multi-component medium, etc.). In this case one has to expect the appearance of a system of several waves.

Let us consider in more detail the case of weak shock waves; in this case the shock wave relations can be linearised.

2. Weak shock waves. Let us write the parameters of the medium behind the surface of the shock wave in the form

$$(n = 1, \ldots, N)$$

$$p_- = p_+ + p', \quad \rho_{n-} = \rho_{n+} + \rho_n', \ \rho_{n-}^\circ = \rho_{n+}^\circ + \rho_n^{\circ\prime}, \quad v_{n-} = v_{n+} + v_n'$$

where p', ρ_n', $\rho_n^{0\prime}$, v_n' are assumed to be so small that their products can be neglected. Substituting these expressions in (1.13), let us neglect the products and squares of small quantities. Then, bearing in mind the relation (1.5), we obtain the following system of $3N + 1$ equations, which are homogeneous with respect to the $3N + 1$ unknowns p', ρ_n', $\rho_n^{0\prime}$, v_n' ($n = 1, \ldots, N$):

$$\rho_n'(D - v_{n+}) = v_n'\rho_{n+} \qquad (n = 1, \ldots, N) \tag{2.1}$$

$$\rho_n'[(D - v_{n+})^2 + p_+/\rho_{n+}^\circ] - 2v_n'\rho_{n+}(D - v_{n+}) + p'\rho_{n+}/\rho_{n+}^\circ -$$

$$- \rho_n^{\circ\prime}p_+\rho_{n+}/\rho_{n+}^{\circ 2} = 0 \tag{2.2}$$

$$p' = a_{n+}^2 \rho_n{}^{\circ\prime} \qquad (a_{n+}{}^2 = \varphi_n{}' (\rho_{n+}{}^{\circ},\ p_0,\ \rho_{0n}{}^{\circ})) \tag{2.3}$$

$$\sum_{n=1}^{N} \frac{1}{\rho_{n+}{}^{\circ}} \left(\rho_n{}' - \frac{\rho_{n+}}{\rho_{n+}{}^{\circ}} \rho_n{}^{\circ\prime} \right) = 0 \tag{2.4}$$

The condition for the existence of non-zero solutions of this system is that its determinant should vanish. This condition determines the values of D corresponding to non-zero solutions. Let us obtain this condition more simply. Let us substitute the value $\rho_n{}^0$, obtained from (2.3), in the equations (2.4) and (2.2).

Then we have

$$\sum_{n=1}^{N} \frac{1}{\rho_{n+}{}^{\circ}} \left(\rho_n{}' - \frac{\rho_{n+}}{\rho_{n+}{}^{\circ} a_{n+}{}^2} p' \right) = 0 \tag{2.5}$$

$$\rho_n{}' [(D - v_{n+})^2 + \lambda_n{}^2] - 2v_n{}'\rho_{n+} (D - v_{n+}) + p'\nu_n\rho_{n+} / \rho_{n+}{}^{\circ} = 0$$

where

$$\lambda_n{}^2 = \frac{p_+}{\rho_{n+}{}^{\circ}}, \qquad \nu_n = 1 - \frac{p_+}{\overset{\circ}{\rho}_{n+} a_{n+}{}^2}$$

From the last equation, using (2.1), we have

$$\rho_n{}' = p' \frac{\rho_{n+}\nu_n}{\rho_{n+}{}^{\circ} [(D - v_{n+})^2 - \lambda_n{}^2]} \qquad (n = 1, \ldots, N) \tag{2.6}$$

Substituting this expression in (2.5), we obtain an equation which is the condition for the existence of non-zero solutions of the system under consideration:

$$\sum_{n=1}^{N} \frac{\rho_{n+}}{\rho_{n+}{}^{\circ 2}} \left[\frac{\nu_n}{(D - v_{n+})^2 - \lambda_n{}^2} - \frac{1}{a_{n+}{}^2} \right] = 0 \tag{2.7}$$

This equation gives, generally speaking, $2N$ values of D, whilst the number of real solutions depends upon the parameters of the medium in front of the surface of discontinuity.

From (2.1) and (2.6) we obtain a relation connecting the pressure and velocity of the n'th component behind the surface of discontinuity:

$$p' = v_n{}' \frac{\rho_{n+}{}^{\circ} [(D - v_{n+})^2 - \lambda_n{}^2]}{\nu_n (D - v_{n+})} \qquad (n = 1, \ldots, N) \tag{2.8}$$

From the relation (2.1) we see that if the condition $D = v_{n+}$ is satisfied for the n'th component $v_n' = 0$. From the foregoing section it is obvious that in this case the discontinuity can be complex.

From the relation (2.8) it follows that, if the relation

$$D - v_{n+} = \pm \lambda_n \tag{2.9}$$

is fulfilled for the n'th component, then $p' = 0$, i.e. the discontinuity is one with equal pressure. But then by virtue of (2.8) the expression (2.9) holds for all the components.

Let us consider in more detail the case of a medium consisting of two components. Then, introducing the notation $D - v_{2+} = y$, $v_{2+} - v_{1+} = \Delta v$, we obtain from (2.7):

$$\Delta v = -y \pm \sqrt{\frac{Ay^2 - E}{Cy^2 - B}} \tag{2.10}$$

where

$$A = \frac{1}{\rho_{1+}^{\circ}}\left(\frac{\rho_{1+}}{\rho_{1+}^{\circ}} + \frac{p_+\rho_{2+}}{\rho_{2+}^{\circ 2}a_{2+}^{2}}\right), \qquad C = \frac{\rho_{2+}}{\rho_{2+}^{\circ 2}a_{2+}^{2}} + \frac{\rho_{1+}}{\rho_{1+}^{\circ 2}a_{1+}^{2}}$$

$$B = \frac{1}{\rho_{2+}^{\circ}}\left(\frac{\rho_{2+}}{\rho_{2+}^{\circ}} + \frac{p_+\rho_{1+}}{\rho_{1+}^{\circ 2}a_{1+}^{2}}\right), \qquad E = \frac{p_+}{\rho_{2+}^{\circ}\rho_{1+}^{\circ}} \tag{2.11}$$

Let us study the function (2.10). Its asymptotes are

$$y = -\Delta v \pm \sqrt{A/C}, \qquad y = \pm\sqrt{B/C}$$

The graph shown in the figure portrays the function (2.10) in the case when the following inequality is satisfied:

$$B/C > E/A \tag{2.12}$$

It is easy to show that this inequality is satisfied in all cases of practical interest.

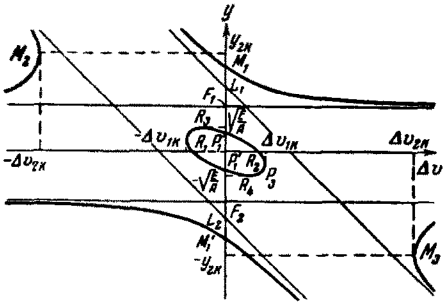

Fig. 1.

In order to determine the values of y corresponding to the case $\Delta v = 0$, let us equate y^2 to the expression under the square root sign in formula (2.10). Then we obtain

$$y^2 = \frac{A + B \pm \sqrt{(A + B)^2 - 4EC}}{2C} \tag{2.13}$$

From (2.12) it follows that $(A + B)^2 > 4EC$ and therefore formula (2.13) gives four real values for the velocity of translation of the weak shock wave (points M_1, M_1', P_1, P_1' in the figure).

We see from the figure that for values of Δv which satisfy the inequalities

$$|\Delta v_{1k}| > |\Delta v| > 0, \qquad |\Delta v| > |\Delta v_{2k}|$$

there are four possible velocities of translation of weak shock waves, whilst in the range

$$|\Delta v_{2k}| > |\Delta v| > |\Delta v_{1k}|$$

there exist only two real values of this velocity. To the values $|\Delta v| = |\Delta v_{1k}|$ and $|\Delta v| = |\Delta v_{2k}|$ there correspond three different values for y.

These critical values for the velocity difference $\pm\Delta v_{1k}$ and $\pm\Delta v_{2k}$ (the points P_2, P_3, M_2, M_3 in the figure) are extremal values of the function (2.10). In order to determine them let us equate to zero the derivative of the function (2.10), thus obtaining

$$\frac{A}{C}\left(\frac{B}{C} - \frac{E}{A}\right)^2 y_k^2 = \left(y_k^2 - \frac{E}{A}\right)\left(y_k^2 - \frac{B}{C}\right)^3$$

It is easy to see that there exist only two real values of y^2 satisfying this equation. Of these, $y_{1k}^2 < E/A$ and $y_{2k} > B/C$, as shown in the figure.

The points R_1, R_2 ($\Delta v = \pm\sqrt{E/B}$, $y = 0$) and R_3, R_4 ($\Delta v = \mp\sqrt{E/A}$, $y = \pm\sqrt{E/A}$) correspond to complex weak discontinuities.

Let us investigate how the velocities of translation of weak shock waves change as a function of the quantitative proportions of the components in the mixture. Let us consider the case of equal velocities of the components in front of the surface of the wave (in particular, the case of propagation of a wave into a stationary mixture). Then from the relation (2.8) for the first and second component we have

$$\frac{v_2'}{v_1'} = \frac{\rho_{1+}{}^\circ \nu_2 [(D - v_+)^2 - \lambda_1^2]}{\rho_{2+}{}^\circ \nu_1 [(D - v_+)^2 - \lambda_2^2]} \tag{2.14}$$

It is not difficult to verify that as the quantity $m = \rho_{1+}/\rho_{1+}{}^0$ varies in the range

$$0 < m < 1 \tag{2.15}$$

the function (2.13) changes monotonically. The limiting values of the function (2.13) corresponding to the transition to a single-component medium are as follows: $y_1^2 = a_{2+}^2$, $y_2^2 = \lambda_1^2$ for $m = 0$, and $y_1^2 = a_{1+}^2$, $y_2^2 = \lambda_2^2$ for $m = 1$.

Accordingly, with the growth of the fraction of the first component in the mixture the value $y_1 = D_1 - v_+$ of the greater velocity of propagation of the wave varies monotonically, from a value equal to the sound velocity in a medium composed entirely of the second component, to the value of the sound velocity in a medium composed entirely of the first component.

The existence of limiting values (when $m \to 0$, $m \to 1$) for the smaller velocity of propagation of the wave $y_2 = D_2 - v_+$ does not indicate, of course, the possibility of existence in a single-component medium of a second velocity of propagation of a disturbance, since the intensity of the wave moving with the velocity D_2 tends to zero with transition to the limiting cases.

In fact, for example, in the case $m \to 0$, i.e. $(D_2 - v_+)^2 \to \lambda_1^2$, we shall have $p' \to 0$, which follows from equation (2.8) for the first component, and then from equation (2.8) for the second component it follows that $v_2' \to 0$. Equations (2.1) and (2.3) in this case give $\rho_2' \to 0$, $\rho_2^{0\prime} \to 0$.

Since for values of m lying in the range (2.15) the quantity $(D_2 - v_+)^2$ satisfies the inequality

$$\lambda_1^2 > (D_2 - v_+)^2 > \lambda_2^2 \quad (\text{or} \quad \lambda_1^2 < (D_2 - v_+)^2 < \lambda_2^2)$$

(in view of its monotonic variation in the range considered), from the relation (2.14) we find that $v_2'/v_1' < 0$. Consequently, if the velocity of one of the components increases as a wave moving with the velocity $D_2 - v_+$, then the velocity of the other component decreases. But then it follows from the relation (2.1) that an increase of the peculiar density of one of the components results in a decrease of the peculiar density of the other component. We notice that a decrease in the peculiar density of a component does not imply a decrease in its true density: when $p' > 0$ we always have $\rho_n^{0\prime} > 0$.

It is obvious that, for a wave propagated with the velocity $D_1 - v_+$, we have $v_2'/v_1' > 0$ (since $(D_1 - v_+)^2 > \lambda_1^2$, $(D_1 - v_+)^2 > \lambda_2^2$).

Using (2.13), let us find the limiting values of the derivative for the case of the upper sign (corresponding to the velocity D_1:

$$\left[\frac{d(y^2)}{dm}\right]_0 = -a_{2+}^4 \frac{\rho_{2+}^\circ}{\rho_{1+}^\circ}\left(\frac{1}{a_{1+}^2} - \frac{1}{a_{2+}^2}\right)\frac{a_{2+}^2 - \lambda_2^2}{a_{2+}^2 - \lambda_1^2} \qquad \text{for } m = 0$$

$$\left[\frac{d(y^2)}{dm}\right]_1 = -a^4_{1+} \frac{\rho_{1+}^\circ}{\rho_{2+}^\circ}\left(\frac{1}{a_{1+}^2} - \frac{1}{a_{2+}^2}\right)\frac{a_{1+}^2 - \lambda_1^2}{a_{1+}^2 - \lambda_2^2} \qquad \text{for } m = 1$$

Suppose that the first component is a gas and the second component a dense, almost incompressible, medium (water, sand etc.). Then $a_{2+} \gg a_{1+}$, $\rho_{2+}^0 \gg \rho_{1+}^0$ and consequently the value of $[d(y^2)/dm]_0$ is very large in absolute value, and moreover $[d(y^2)/dm]_0 \gg [d(y^2)/dm]_1$. This means that, with the occurrence in the dense medium of even an insignificant quantity of gas (air), the value $D_1 - v_+$ of the velocity of propagation of waves very quickly drops by comparison with the velocity of sound in the dense medium when free from contamination by the gas; for a comparatively small volume of entrained gas in the dense medium the value of the velocity of propagation of the wave in the given two-component medium approaches the value of the velocity of sound in the gas. If the medium is a mixture of two dense components, then the change in the quantity $D_1 - v_+$ from the value a_{2+} to the value a_{1+} takes place more uniformly.

For studying the dependence of the quantity $y^2 = (D - v_{2+})^2$ on the quantitative composition of a two-component mixture with arbitrary values of Δv it is sufficient to consider the deformation of the graph in the figure, as the quantity m varies in the range (2.15).

It is not difficult to show that the characteristic points indicated in the figure move monotonically, and that in the limiting cases the graph degenerates into straight lines, corresponding to the limiting positions of the asymptotes.

Let us consider now the case of an N-component medium. From equation (2.7), introducing the notation

$$z_n = y + v_{N+} - v_{n+} \qquad (y = D - v_{N+}) \qquad (n = 1, \ldots, N) \tag{2.16}$$

we obtain

$$y = \pm\left(\frac{l_N}{\Phi} + \lambda_N^2\right)^{1/2} \tag{2.17}$$

where

$$\Phi = \sum_{n=1}^{N} \frac{\rho_{n+}}{\rho_{n+}{}^{\circ 2} a^2{}_{n+}} - \sum_{n=1}^{N-1} \frac{l_n}{z_n{}^2 - \lambda_n{}^2}, \qquad l_n = \frac{\rho_{n+}}{\rho_{n+}{}^{\circ 2}} \nu_n \qquad (n = 1, \ldots, N)$$

Equations (2.16) and (2.17) describe a straight line and a surface respectively in N-dimensional space, defined by the system of coordinates $(z_1, \ldots, z_{N-1}, y)$. Their point of intersection gives the values of $z_1, \ldots, z_{N-1}, y$, for which the velocity D of translation of the wave is determined.

It is easy to study the surface (2.17) by dissecting it with planes passing through the y-axis. In particular, it is not difficult to show that the existence of the maximum number of waves ($2N$) is possible for sufficiently small and for sufficiently large values of the differences of the velocities of the components.

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